Closed Graph Theorem

Def^n Graph : Let X & Y be any non-empty sets and let $f: X \to Y$ be a mapping with domain X and range in Y. Then the graph of f is defined as a subset of $X \times Y$ which consists of all ordered pairs of the form $(x, f(x))$
$$\left[\begin{matrix} \in & \in \\ X & Y \end{matrix}\right]$$

Graph of f is denoted by f_G.

Closed Linear Transformation

Let $N, N' \to$ NLS. Let D be a subspace of N. Then a L.T. $\to$

$T: D \to N'$ is said to be closed iff $x_n \in D$ s.t. $x_n \to x$ & $T(x_n) \to y$ imply that $x \in D$ and $y = T(x)$

Th^m Let N & N' be NLS and D a subspace of N. Then a L.T. $T: D \to N'$ is closed iff its graph T_G is closed.

Pf → First we assume that T is a closed L.T. We have to show that its graph T_G is closed i.e. T_G contains all its limit points.

Let (x, y) be any limit point of T_G.

Then $\exists$ a seq^n of points in T_G, $\langle x_n, T(x_n)\rangle$

where $x_n \in D$, converging to (x, y).

Now,

$$\langle x_n, T(x_n)\rangle \to (x, y)$$

$$\Rightarrow \|(x_n, T(x_n)) - (x, y)\| \to 0$$

$$\Rightarrow \|(x_n - x), (T(x_n) - y)\| \to 0$$

$$\Rightarrow \|x_n - x\| + \|T(x_n) - y\| \to 0$$

$\because \|(x, y)\| = \|x\| + \|y\|$

$$\Rightarrow \|x_n - x\| \to 0 \text{ and } \|T(x_n) - y\| \to 0$$

$$\Rightarrow x_n \to x \text{ and } T(x_n) \to y$$

$$\Rightarrow x \in D \text{ and } T(x) = y$$

$\because T$ is closed

$$\Rightarrow (x, y) \in T_G \text{, by def}^n \text{ of graph}$$

$$\Rightarrow T_G \text{ is closed}$$

Converse: Let T_G be closed.

To show that T is a closed L.T.

Let $x_n \in D$, $x_n \to x$ and $T(x_n) \to y$.

Then (x, y) is an adherent point of T_G so that $(x, y) \in \overline{T_G}$. But since T_G is closed, $\therefore \overline{T_G} = T_G$

Hence $(x, y) \in T_G$ and $\therefore$ by def^n of T_G, we have,

$$x \in D,\ y = T(x)$$

$\therefore T$ is a closed L.T.

(Proved)

The Closed Graph Theorem

Let B & B' be Banach spaces and let T be a L.T. of B into B'. Then T is a continuous mapping iff its graph is closed

Pf:- Let T be continuous and let T_G be the graph of T. To show that T_G is closed. For this we need to show that $\overline{T_G} = T_G$. Since $T_G \subset \overline{T_G}$, $\therefore$ we only need to show that $\overline{T_G} \subset T_G$.

Let $(x, y) \in \overline{T_G}$. Then (x, y) is an adherent point of T_G. $\therefore \exists$ a seqn $\langle x_n, T(x_n) \rangle$ in T_G s.t.

$$(x_n, T(x_n)) \to (x, y)$$

$$\text{i.e. } x_n \to x \quad \& \quad T(x_n) \to y$$

But, since T is continuous, $\therefore$

$$x_n \to x \Rightarrow T(x_n) \to T(x)$$

$$\Rightarrow y = T(x)$$

$$\therefore (x, y) = (x, T(x)) \in T_G \Rightarrow \overline{T_G} \subset T_G$$

$$\therefore \overline{T_G} = T_G$$

$$\Rightarrow T_G \text{ is closed.}$$

Converse Let T_G be closed. To show that T is continuous

Consider the norm $\|\cdot\|_1$ defined by -

$$\|x\|_1 = \|x\| + \|T(x)\|$$

Let $\|\cdot\|_1$ be defined on B and we denote this linear space by B_1.

Then $\|T(x)\| \leq \|x\| + \|T(x)\| = \|x\|_1$

i.e. $\|T(x)\| \leq \|x\|_1$

$\Rightarrow$ T is bdd.

$\Rightarrow$ T is continuous on $\|\cdot\|_1$

So if we show that B_1 & B have same topology i.e. Both are homeomorphic then T will be bdd on B as well.

Consider the identity map $\rightarrow$

$$I : B_1 \rightarrow B$$

$$\text{s.t. } I(x) = x \quad \forall x \in B_1$$

Clearly I is 1-1 & onto.

Also, $\|I(x)\| = \|x\| \leq \|x\| + \|T(x)\| = \|x\|_1$

$\Rightarrow$ I is bdd. $\Rightarrow$ I is cont.

Now if we show that B_1 is complete then I will be an homeomorphism and so B_1 & B will be homeomorphic & then T is bdd. & continuous.

To show that B_1 is complete, consider a Cauchy seq^n $\langle x_n \rangle$ in B_1. Then $\rightarrow$ ~~$\|x_n - x_m\|$~~

$$\|x_n - x_m\|_1 \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

$$\Rightarrow \|x_n - x_m\| + \|T(x_n - x_m)\| \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

$$\because \|x\|_1 = \|x\| + \|T(x)\|$$

$\Rightarrow \|x_n - x_m\| \rightarrow 0$ and $\|T(x_n) - T(x_m)\| \rightarrow 0$ as $m, n \rightarrow \infty$

$\Rightarrow \langle x_n \rangle$ is a Cauchy seq^n in B and $\langle T(x_n) \rangle$ is a Cauchy seq^n in B'.

Since B & B' are Banach spaces,

$\therefore$ $x_n \to x \in B$ & $T(x_n) \to y \in B'$ $\to (1)$

Since graph of T i.e. T_G is closed,

$\therefore$ $(x, y) \in T_G \Rightarrow y = T(x)$

Now $\to$

$$\|x_n - x\|_1 = \|x_n - x\| + \|T(x_n - x)\|$$

$$= \|x_n - x\| + \|T(x_n) - T(x)\|$$

$$= \|x_n - x\| + \|T(x_n) - y\| \quad ; \because y = T(x)$$

$$\to 0$$

as $x_n \to x$ & $T(x_n) \to y$ from (1)

$\therefore$ The Cauchy seq^n $\langle x_n \rangle$ in B_1 converges to x in B_1 $\Rightarrow$ B_1 is complete

$\Rightarrow$ T is bdd.

$\Rightarrow$ T is continuous.

(Proved)